

QUARTERLY PROGRESS REPORT

For Period

1 January through 31 March 1966

INVESTIGATION OF OPTIMIZATION OF  
ATTITUDE CONTROL SYSTEMS

For

Jet Propulsion Laboratory

4800 Oak Grove Drive

GPO PRICE \$ \_\_\_\_\_

Pasadena, California

CFSTI PRICE(S) \$ \_\_\_\_\_

on

Hard copy (HC) \$ 2.00

Microfiche (MF) 1.50

Contract No. 950670

FF 653 July 65

School of Electrical Engineering

Purdue University

Lafayette, Indiana 47907

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FACILITY FORM 602

N66 24591

(ACCESSION NUMBER)

43

(PAGES)

PR 74627

(NASA CR OR TMX OR AD NUMBER)

(THRU)

(CODE)

10

(CATEGORY)

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This work was performed for the Jet Propulsion Laboratory, California Institute of Technology, sponsored by the National Aeronautics and Space Administration under Contract NAS7-100.

## TABLE OF CONTENTS

	<u>PAGE</u>
PART A - GENERAL DISCUSSION. . . . .	1
I. Introduction . . . . .	1
II. Progress During Reporting Period . . . . .	1
2-1 Coordination Meetings . . . . .	1
2-2 Technical Progress. . . . .	2
III. Professional Contributors. . . . .	3
PART B - OPTIMAL CONTROL OF A PARTICULAR SPACE VEHICLE . . . . .	4
I. Introduction . . . . .	4
II. Preliminary Results. . . . .	5
III. Other Formulations of the Particular Vehicle Control . . . . .	9
3-1 Controlled State Variables Including Displacement . . . . .	9
3-2 Gyro Transfer Function Including Inertia Term . . . . .	10
IV. Plan of Future Work. . . . .	11
PART C - OPTIMAL CONTROL IN BOUNDED PHASE-COORDINATE PROCESS . . . . .	12
I. Introduction . . . . .	12
II. Analysis of Optimal Control of a Harmonic Oscillator . . . . .	13
III. The Extremal Controls. . . . .	15
IV. Plan of Future Work. . . . .	19
Chart I. . . . .	20a
Chart II . . . . .	20b
PART D - STOCHASTIC OPTIMAL CONTROL. . . . .	21
I. Introduction . . . . .	21
II. Formulation of Antenna Pointing Problem. . . . .	21
III. Summary of Technical Progress. . . . .	25
IV. Plan of Future Work. . . . .	27
APPENDIX . . . . .	28
REFERENCES . . . . .	33

## PART A

### GENERAL DISCUSSION

#### 1. INTRODUCTION

This is the seventh quarterly progress report submitted in accordance with the provisions of Contract No. 950670, "Investigation of Optimization of Attitude Control Systems." It covers the period January 1, 1966 through March 31, 1966.

This report is in four parts. The first part summarizes the progress during the reporting period. The technical discussions are given in Parts B, C, and D, in which the plans of future work are also included.

#### 2. PROGRESS DURING REPORTING PERIOD

##### 2-1 Coordination Meetings

A series of meetings was held at Purdue University on March 9 through March 11, 1966. Those present at the meetings included: A. E. Cherniack of Jet Propulsion Laboratory, Pasadena, California, and J. Y. S. Luh, G. E. O'Connor and J. S. Shafran of Purdue University. The discussions brought out the following:

(a) The antenna pointing problem should be treated as a steady state problem or its equivalent.

(b) The optimal control of a particular space landing vehicle should be investigated with respect to the application of the control theory of the bounded phase-coordinate systems.

(c) The study of a harmonic oscillator with bounded amplitude and bounded rate control should not be abandoned since the investigation was almost completed at the meeting date.

(d) JPL will initiate an amendment to Contract No. 950670 to revise the STATEMENT OF WORK in accordance with the current research tasks, which were agreed upon by both JPL and Purdue.

(e), The research fund for the third year period is not guaranteed. The research, however, will be carried on until the free balance of the account is exhausted. It is understood that the continuity of the research effort will be interrupted should JPL decide to fund the third year period at a later date.

#### 2-2 Technical Progress

A preliminary study of the optimal control of a particular space vehicle was completed. Based on the switching curve and the boundary of the controllable region, a closed-loop optimal controller was synthesized for the system in which the gyro inertia can be ignored and the displacement of the vehicle is not specified. The control is of a linear type if the gyro torque is saturated, and is of the "bang-bang" type otherwise. The solution in detail, as well as the plan of future work, is given in Part B.

The investigation of a harmonic oscillator with bounded amplitude and bounded rate was also completed. This study is a continuation of the previous work [3]. The result gives an insight into the form of the extremal control function for a system having a pair of purely imaginary characteristic roots. The problem is more complicated than the unstable booster problem [3]. The control variable is found to enter upon and exit from its bound as often as the time duration permits, which is a natural result of the oscillatory behavior of the system. Part C gives a complete presentation of the findings.

The study of the antenna pointing problem is being continued. The optimal control problem is reformulated in such a manner that the pointing direction is kept within an accepted region with maximum probability all the time. Essentially, the optimal controller minimizes the error rate of transmission of information during the entire flight journey of the space vehicle by enforcing the antenna to point in a proper direction. Based on this criterion, the optimal control problem is divided into two parts, viz., the determination of the probability and the maximization (or minimization) of it. During the reporting period, only the first part was studied. The technical discussion and the plan of future work are given in Part D.

### 3. PROFESSIONAL CONTRIBUTORS

Professional personnel contributing to the progress during the reporting period are as follows:

J. Y. S. Luh, Principal Investigator

G. E. O'Connor, Staff Researcher

J. S. Shafran, Staff Researcher

## PART B

### OPTIMAL CONTROL OF A PARTICULAR SPACE VEHICLE

#### 1. INTRODUCTION

During the coordination meeting at Purdue on March 10, 1966, the control problem of a particular space vehicle was discussed. The block diagram for this particular system is shown in Figure 1. When the system is operated in the linear regions of both saturation elements, the over-all closed loop transfer function is

$$\frac{\theta(s)}{r(s)} = \frac{1}{Js^2 + [K_1K_2s + K_1K_3] G(s)} \quad (1)$$

where  $G(s)$  is the gyro transfer function. Since  $G(s)$  always acts as a lag network, the system is unstable. With  $G(s) = 1/s(Is + D)$ , equation (1) becomes

$$\frac{\theta(s)}{r(s)} = \frac{s(Is + D)}{Js^3(Is + D) + K_1K_2s + K_1K_3}.$$

Because of instability, the system operates in a saturation mode (for both saturation elements). Since the two saturation elements are in different loops, an analytic solution based on classic control theory is quite involved.

The control problem, however, can be simplified to some extent if a nonlinear controller is introduced and synthesized on the open-loop point of view. To be specific, the system block diagram is redrawn as in Figure 2. The original input  $r$ , which represents either a command or a disturbance, can be interpreted as a set of initial conditions  $\theta_0$ ,  $\dot{\theta}_0$ , and  $\ddot{\theta}_0$ . The first objective is to find an open-loop control law  $u$ , which depends on initial conditions only, to steer the system

to a desired set of terminal conditions  $\theta_d$ ,  $\dot{\theta}_d$ , and  $\ddot{\theta}_d$ . Based on this result, a controller  $\Psi$  is synthesized and the system loop is then closed. If, at the same time, a minimization (or maximization) of a prescribed performance index is desired, then the problem becomes an optimal control problem. To facilitate the mathematical manipulation, assuming the system is operated in the linear regions, then with the absence of  $\Psi$ ,

$$\frac{s\theta(s)}{-u(s)} = \frac{K_1 G(s)}{Js + K_1 K_2 G(s)} \quad (2)$$

In the following sections, a discussion on the optimal control problem with saturations for various  $G(s)$  is presented. Section 2 discusses the problem where the inertia of the gyro is negligible. It turns out to be a bounded phase-coordinate control problem. With the performance index being minimal time, the problem is solved for the case of unspecified  $\theta$ . A numerical example is included for illustration. Section 3 defines the problems that assume different forms of gyro transfer functions. Each of the problems can be treated as an optimal control problem in the bounded phase-coordinate system. Section 4 outlines the plan of future study of these problems with various performance indices.

## 2. PRELIMINARY RESULTS

If the inertia of the gyro shown in Figure 2 is negligible, i.e.,  $G(s) = 1/Ds$ , then the over-all transfer function given in equation (2) becomes

$$\frac{s\theta(s)}{-u(s)} = \frac{K_1}{JDs^2 + K_1 K_2}$$

For simplicity, consider a normalized transfer function of  $1/(s^2 + 1)$ .



An equivalent system can be described by a second order differential system

$$\dot{x} = Ax(t) + b u(t) \quad (3)$$

where

$$x(t) = \begin{bmatrix} \ddot{\theta}(t) \\ \dot{\theta}(t) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

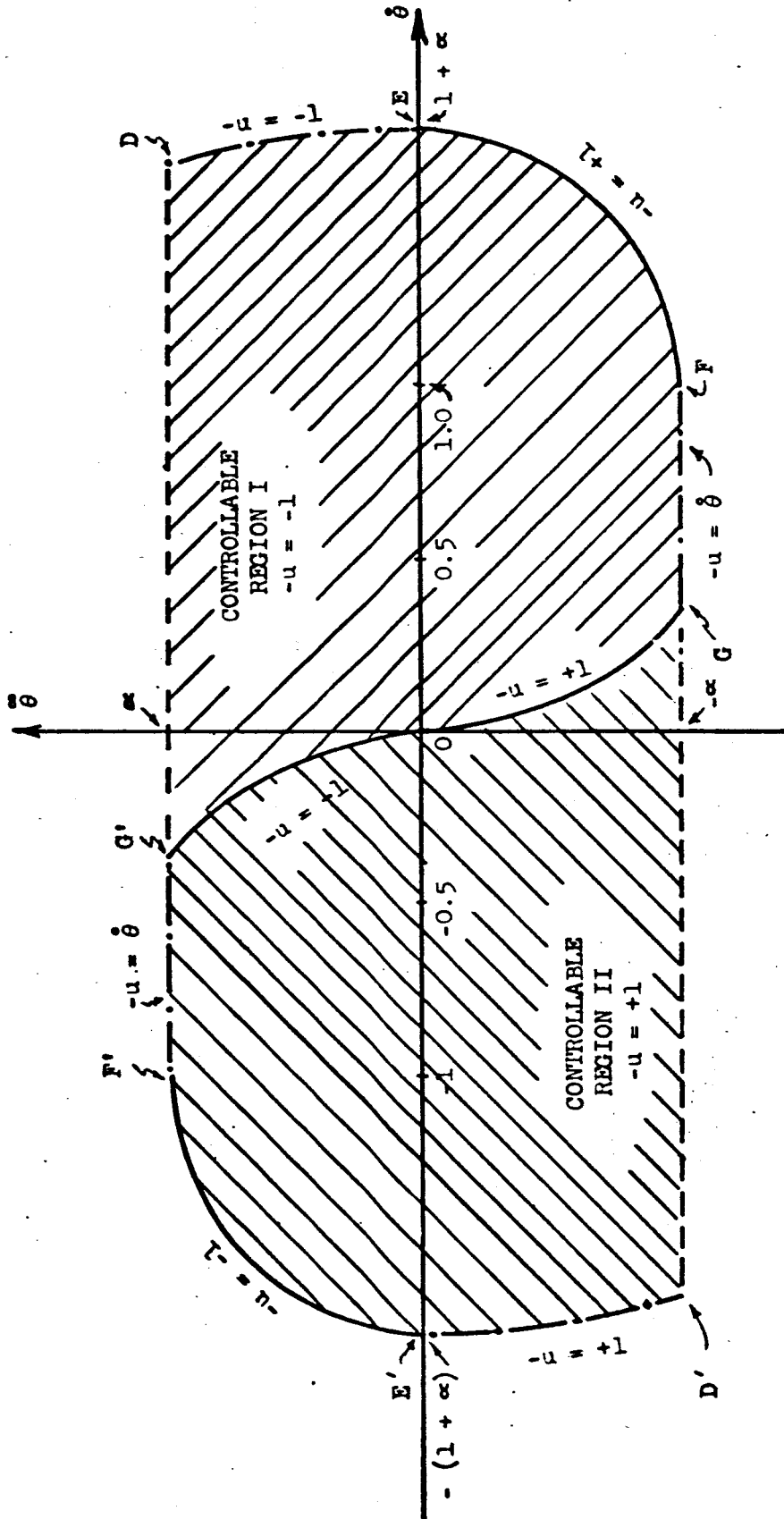
To include the saturation effect, as shown in Figure 2, the constraints  $|\ddot{\theta}(t)| \leq \alpha$  and  $|u(t)| \leq 1$  (normalized) are required. Under these conditions, it is desired to determine a control law  $u(t)$  which steers the system (3) from an initial state  $x(0) = \begin{bmatrix} \ddot{\theta}_0 \\ \dot{\theta}_0 \end{bmatrix}$  to the prescribed terminal state  $x(T) = \begin{bmatrix} \ddot{\theta}_d \\ \dot{\theta}_d \end{bmatrix}$  with a minimal performance index. If  $\ddot{\theta}_d = \dot{\theta}_d = 0$  and the performance index is minimal time  $T$ , it becomes a time-optimal regulator problem in the bounded phase-coordinate system. For  $|\ddot{\theta}(t)| \leq \alpha \leq 1$ , the problem is solved by Russell [1] which gives a "bang-bang" type of control when the trajectory stays inside the bound, and a linear control when it stays on the boundary. The switching curve and the controllable region are shown in Figure 3. The boundary of the controllable region and the switching curve consist of circular arcs as follows:

Arc DE : center at  $(-1, 0)$ , radius  $2 + \alpha$ ;

Arc EF : center at  $(1, 0)$ , radius  $\alpha$ ;

Arc GO: center at  $(1, 0)$ , radius 1.

Arcs D'E', etc. are their symmetric counterparts. With the switching curve so defined, the control law is given as



$$0 < \alpha \leq 1$$

$$|u| \leq 1, |\dot{\theta}| \leq \alpha$$

FIGURE 3

$$-u(t) = \begin{cases} -1 & \text{if } (\dot{\theta}, \ddot{\theta}) \in \text{interior of region I, or} \\ & (\dot{\theta}, \ddot{\theta}) \text{ on arc DE or arc E'F' or arc G'O,} \\ +1 & \text{if } (\dot{\theta}, \ddot{\theta}) \in \text{interior of region II, or} \\ & (\dot{\theta}, \ddot{\theta}) \text{ on arc EF or arc D'E' or arc GO,} \\ \dot{\theta} & \text{if } (\dot{\theta}, \ddot{\theta}) \text{ on chord FG or chord F'G'.} \end{cases}$$

Thus the nonlinear controller  $\Psi$  can be constructed by means of storing the control law in some on-line device.

The controllable region is determined for the case of minimal time and  $\dot{\theta}_d = \ddot{\theta}_d = 0$ . Therefore any initial state, which is outside of the controllable region, cannot be brought back to the origin in a finite time interval under the constraints  $|\ddot{\theta}(t)| \leq \alpha$ ,  $0 < \alpha \leq 1$ , and  $|u(t)| \leq 1$ .

### 3. OTHER FORMULATIONS OF THE PARTICULAR VEHICLE CONTROL

#### 3-1 Controlled State Variables Including Displacement

In the above discussion, the displacement  $\theta$  was left free. If  $\theta$  also has its desired terminal value, the differential system (3) is augmented to a third order system

$$\dot{\bar{x}} = \bar{A} \bar{x}(t) + \bar{b} u(t)$$

where

$$\bar{x}(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \bar{b} = -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and

$$|\ddot{\theta}(t)| \leq \alpha, \quad |u(t)| \leq 1.$$

The problem is to find a controller  $u(t)$  to bring an initial state  $\bar{x}(0) = \bar{x}_0$  to the terminal state  $\bar{x}(T) = \bar{x}_d$  with some prescribed performance index being minimized at the same time.

Although this problem is more general than the simple bounded phase-coordinate control discussed in the preceding section, the inertia term of the gyro is ignored. When the time interval of the operation becomes short, which is exactly the objective of the time-optimal control system, the inertia term may result in an undesirable time lag so that it cannot be ignored in synthesizing the controller. This problem will be formulated next.

### 3-2 Gyro Transfer Function Including Inertia Term

With  $G(s) = 1/s(Is + D)$ , the over-all transfer function given in equation (2) becomes

$$\frac{s\theta(s)}{-u(s)} = \frac{K_1}{JIs^3 + JDs^2 + K_1K_2}.$$

For a normalized transfer function of  $1/(s^3 + \beta s^2 + 1)$ , the equivalent differential system can be written as

$$\dot{\hat{x}} = \hat{A}\hat{x}(t) + \hat{b} u(t)$$

where

$$\hat{x}(t) = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \\ \ddot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -\beta \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

and

$$|\ddot{\theta}(t)| \leq \alpha, \quad |u(t)| \leq 1.$$

As before, the controller  $u(t)$  is required to drive  $\hat{x}(0) = \hat{x}_0$  to  $\hat{x}(T) = \hat{x}_d$  with a performance index minimal. This problem and the one formulated in Section 3-1 will be investigated as outlined in the following section.

#### 4. PLAN OF FUTURE WORK

The time-optimal regulator problem without the gyro inertia term (Section 3-1) will be investigated first. Gamkrelidze's [2] necessary condition, together with Russell's [1] sufficiency condition, will be applied. It is expected that the switching surface and the controllable region can be determined in the phase-coordinate space. By storing this information in some on-line device, a closed-loop time-optimal regulator is feasible.

The next step is to study the minimum energy control problem for the same system. At the present time it is not certain if a switching surface can be found explicitly. In any event, an explicit time function for the controller can be determined by the method of maximizing the projection of the set of attainability onto the adjoint vector [3]. The storage of the time function makes the sample-and-hold type closed-loop optimal control possible.

The controller so determined will then be simulated on a digital computer and the results evaluated. The same steps of investigation will be applied to the system which includes the gyro inertia term. A careful comparison of these results against the existing controller will determine the merit of this investigation.

## PART C

### OPTIMAL CONTROL IN BOUNDED PHASE-COORDINATE PROCESS

#### 1. INTRODUCTION

A method of determining the optimal control in the bounded phase-coordinate system was developed and presented in the previous report [3]. An application of the method to the time-optimal control of an unstable booster with actuator position and rate limits was also given there. In this report, an extremal control of an undamped oscillatory plant (or, a harmonic oscillator) with actuator position and rate limits is presented. The same method is utilized in this application, which is based on the "backing out of the target" procedure. As a first attempt, the ratio of control amplitude limit to the control rate limit is selected as one-fifth of the period of the oscillation. This selection allows the extremal control to enter upon and exit from its bound once every half cycle of oscillation.

In the following sections, the extremal control for a harmonic oscillator is presented. The detailed derivation is also given. Section 2 defines the problem and then analyzes it. Section 3 determines the extremal control by means of the maximization of the projection of the state vector onto a unit adjoint vector for every fixed terminal time. The procedure follows the general pattern as that used in the unstable booster problem, and hence the general theory is omitted in this report. Section 4 gives an outline for the future work.

## 2. ANALYSIS OF OPTIMAL CONTROL OF A HARMONIC OSCILLATOR

A harmonic oscillator is described by a second order differential system

$$\dot{\hat{x}} = \hat{A}x + \hat{b} u(t) \quad (1)$$

where

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix}, \quad \hat{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The problem is to determine an admissible control  $u(t)$  on  $[0, t_1]$  which steers the system (1) from a given initial state  $\hat{x}(0) = \hat{x}_0$  to  $\hat{x}(t_1) = 0$  such that

$$|u(t)| \leq 1, \quad |\dot{u}(t)| \leq 2.5/\pi$$

on  $[0, t_1]$  and  $t_1$  is minimal.

To formulate the problem as a bounded phase-coordinate control problem augment the system by letting  $x_3 = u(t)$ . Furthermore, define  $\tau = -t$  for the purpose of "backing out of the target," then the system (1) becomes

$$dx/d\tau = -Ax(\tau) - b v(\tau) \quad (2)$$

with  $x(0) = 0$  where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \dot{x}_1 \\ u(\tau) \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$v(\tau) = du/d\tau.$$

By the variation of parameters formula, the solution is

$$x(\tau) = \begin{bmatrix} \int_0^\tau [\cos(\tau-s) - 1] v(s) ds \\ \int_0^\tau \sin(\tau-s) v(s) ds \\ -\int_0^\tau v(s) ds \end{bmatrix} \quad (3)$$

where  $|v(s)| \leq 2.5/\pi$  is admissible on  $[0, \tau]$ . To represent the extremal  $v$  as a multiple of the signum of an adjoint solution for the bounded phase-coordinate control problem, the adjoint system must be modified. Consequently, a "total adjoint vector" must satisfy the relation

$$dp/d\tau = \begin{cases} A'p(\tau), & \text{if } |x_3(\tau)| < 1, \\ \tilde{A}'p(\tau), & \text{if } |x_3(\tau)| = 1, \end{cases} \quad (4)$$

where  $A' =$  transpose of  $A$

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus

$$v(\tau) = 2.5 \operatorname{sgn} [p(\tau)' (-b)]/\pi$$

or

$$-v(\tau) = 2.5 \operatorname{sgn} [p_3(\tau)]/\pi$$

in which

$$\operatorname{sgn} p_3 = \begin{cases} +1, & \text{if } p_3 > 0, \\ 0, & \text{if } p_3 = 0, \\ -1, & \text{if } p_3 < 0; \end{cases}$$

and  $p_3(\tau)$  is allowed certain jump discontinuities at endpoints of intervals where  $|x_3(\tau)| = 1$ , and  $p_3(\tau) = 0$  when  $|x_3(\tau)| = 1$ . As a result, the solution of the system (4) is

$$p_1(\tau) = p_1(0) \cos \tau + p_2(0) \sin \tau$$

$$p_2(\tau) = -p_1(0) \sin \tau + p_2(0) \cos \tau$$

$$p_3(\tau) = \begin{cases} -p_1(0) \cos \tau - p_2(0) \sin \tau + k, & \text{if } |x_3(\tau)| < 1 \\ 0, & \text{if } |x_3(\tau)| = 1 \end{cases}$$

where the value of constant  $k$  in  $p_3(\tau)$  depends on  $p_1(0)$ ,  $p_2(0)$  and the interval in which  $|x_3(\tau)| < 1$ .

### 3. THE EXTREMAL CONTROLS

Let the unit adjoint vector at time  $\tau$  be

$$\eta = \begin{bmatrix} \cos \theta \cos \varphi \\ \sin \theta \cos \varphi \\ \sin \varphi \end{bmatrix}, \quad |\theta| \leq \pi, \quad |\varphi| < \pi/2,$$

then the projection of  $x(\tau)$  onto  $\eta$  is

$$P = \cos \varphi \int_0^{\tau} f(s; t, \theta, \varphi) [-v(s)] ds$$

where

$$f(s; t, \theta, \varphi) = \cos \theta - \cos(\tau - \theta - s) + \tan \varphi$$

Since

$$f(s; t, \theta, \varphi) = -f(s; t, \pi + \theta, -\varphi)$$

it suffices to consider only half of the range of  $\theta$ . Choose  $-\pi \leq \theta \leq 0$  for convenience. Then for a fixed  $\tau$ , a fixed  $\theta$ , and a fixed  $\varphi$ ,  $f(s; t, \theta, \varphi)$  can be sketched on the interval  $0 \leq s \leq \tau$ .

To determine the form of extremal  $v(s)$  that maximizes  $P$ , the method of inspection that was employed in the unstable booster problem can be used. A typical case is shown in Figure 4. The ranges are  $-4\pi/5 \leq \theta \leq -3\pi/5$ ,  $11\pi/5 < \tau - \theta < 14\pi/5$ . The form of extremal  $v(s)$  is:

$$(a) \text{ for } \pi/2 > \varphi \geq \tan^{-1} [\cos(6\pi/5 + \theta) - \cos \theta] \geq 0$$

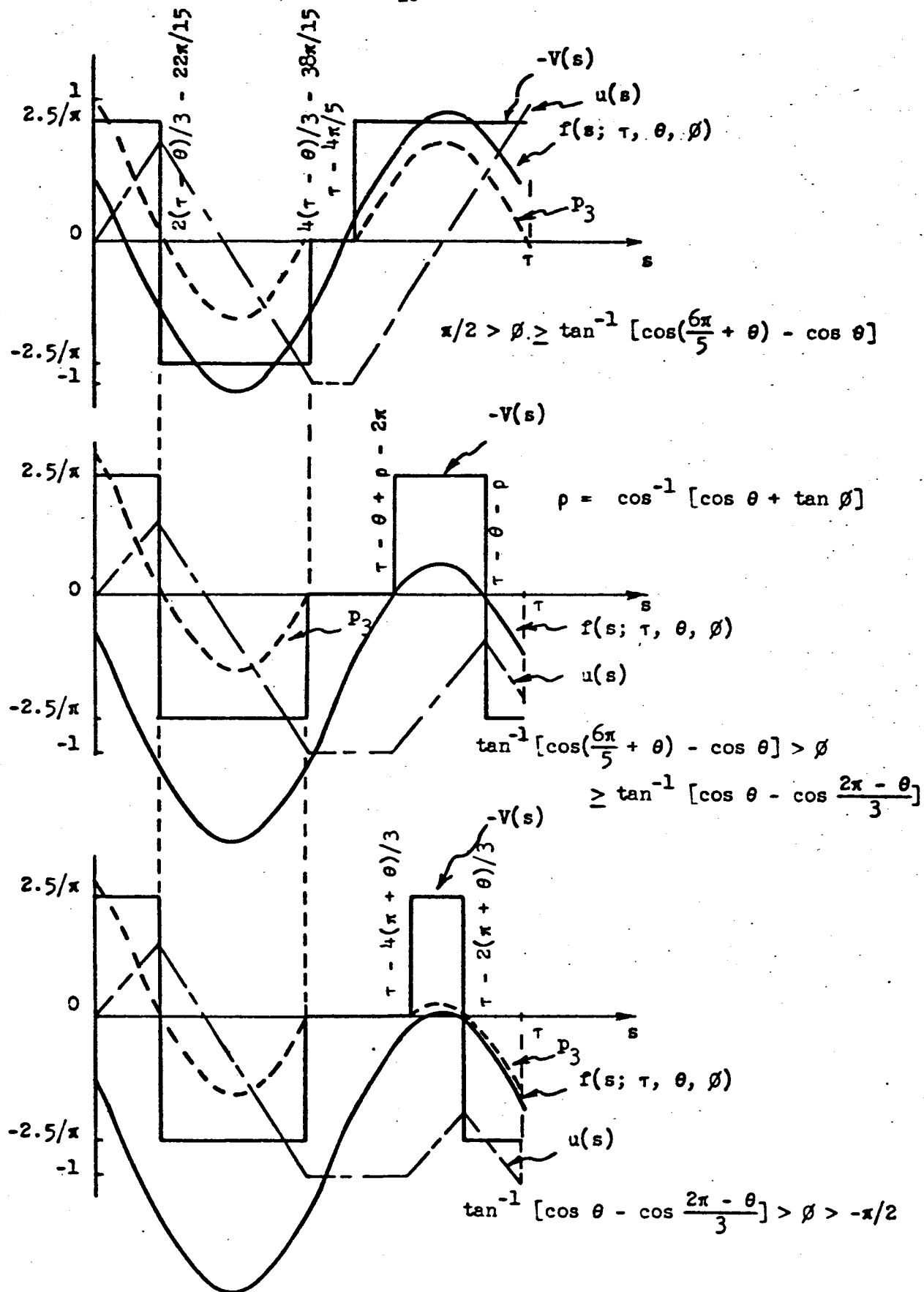


FIGURE 4

$$-v(s) = \begin{cases} 2.5/\pi, & \text{if } 0 < s < 2(\tau - \theta)/3 - 22\pi/15 = s_1 \\ -2.5/\pi, & \text{if } s_1 < s < 4(\tau - \theta)/3 - 38\pi/15 = \tau_1 \\ 0, & \text{if } \tau_1 < s < \tau - 4\pi/5 = \tau_2 \\ 2.5/\pi, & \text{if } \tau_2 < s < \tau, \end{cases}$$

or

$$(b) \text{ for } \tan^{-1} [\cos(6\pi/5 + \theta) - \cos \theta] > \emptyset$$

$$\geq -\tan^{-1} [\cos \theta - \cos((2\pi - \theta)/3)]$$

$$-v(s) = \begin{cases} 2.5/\pi, & \text{if } 0 < s < 2(\tau - \theta)/3 - 22\pi/15 = s_1 \\ -2.5/\pi, & \text{if } s_1 < s < 4(\tau - \theta)/3 - 38\pi/15 = \tau_1 \\ 0, & \text{if } \tau_1 < s < \tau - \theta + \cos^{-1} [\cos \theta + \tan \emptyset] - 2\pi = \tau_2 \\ 2.5/\pi, & \text{if } \tau_2 < s < \tau - \theta - \cos^{-1} [\cos \theta + \tan \emptyset] = s_2 \\ -2.5/\pi, & \text{if } s_2 < s < \tau, \end{cases}$$

or

$$(c) \text{ for } -\tan^{-1} [\cos \theta - \cos((2\pi - \theta)/3)] > \emptyset > -\pi/2$$

$$-v(s) = \begin{cases} 2.5/\pi, & \text{if } 0 < s < 2(\tau - \theta)/3 - 22\pi/15 = s_1 \\ -2.5/\pi, & \text{if } s_1 < s < 4(\tau - \theta)/3 - 38\pi/15 = \tau_1 \\ 0, & \text{if } \tau_1 < s < \tau - 4(\pi + \theta)/3 = \tau_2 \\ 2.5/\pi, & \text{if } \tau_2 < s < \tau - 2(\pi + \theta)/3 = s_2 \\ -2.5/\pi, & \text{if } s_2 < s < \tau. \end{cases}$$

The same procedure was carried out for all possible ranges of  $\theta$  and  $\tau$ . It was found that the extremal  $v(s)$  reaches zero and leaves zero as often as the length of  $\tau$  permits. Denote the time  $s$  at which such events occur by  $\tau_i$ ,  $i = 1, 2, \dots, 2N$ , and let  $\tau_0 = 0$ ,  $\tau_{2N+1} = \tau$ .

Furthermore, let

$$|u(s)| < 1, \text{ if } \tau_{2i} \leq s < \tau_{2i+1}, i = 0, 1, \dots, N;$$

and

$$|u(s)| = 1, \text{ if } \tau_{2j+1} \leq s < \tau_{2j+2}, j = 0, \dots, N-1.$$

Then

$$dp_3/ds = d\Psi_3/ds \text{ for } \tau_{2i} \leq s < \tau_{2i+1}, i = 0, \dots, N;$$

and

$$p_3(s) = 0 \text{ for } \tau_{2j+1} \leq s < \tau_{2j+2}, j = 0, \dots, N-1;$$

where  $\Psi_3$  is a component of  $\Psi$  satisfying  $d\Psi/d\tau = A'\Psi$ . As indicated in the previous report [3], choosing

$$p_3(s) = \Psi_3(s) - \Psi_3(\tau_{2i+1}) \text{ for } \tau_{2i} \leq s < \tau_{2i+1}, i = 0, \dots, N,$$

yields  $p_3(s)$  being zero and continuous at  $\tau_{2i+1}$ , and consequently the jump conditions must be satisfied at  $\tau_{2i}$ ,  $i = 0, \dots, N-1$ . Since  $\eta$  is the unit adjoint vector at time  $\tau$ , thus  $p_3(\tau) = \Psi_3(\tau) = \eta_3 = \sin \phi$ .

Therefore

$$\frac{p_3(s; \tau, \theta, \phi)}{\cos \phi} = \begin{cases} \cos(\tau - \tau_{2i+1} - \theta) - \cos(\tau - s - \theta), & \text{if } \tau_{2i} \leq s < \tau_{2i+1}, \\ 0, & \text{if } \tau_{2i+1} \leq s < \tau_{2i+2}, \\ \cos \theta - \cos(\tau - \theta - s) + \tan \phi \delta (\tan \phi_0 - \tan \phi), & \text{if } \tau_{2N} \leq s < \tau, \end{cases}$$

where

$$i = 0, 1, \dots, N-1$$

$$\delta = \begin{cases} 0, & \text{if } |x_3(\tau)| = |u(\tau)| < 1 \\ 1, & \text{if } |x_3(\tau)| = |u(\tau)| = 1, \end{cases}$$

and

$\phi_0$  = direction limit for  $\eta$  at which  $p_3$  has a jump discontinuity

( $\phi_0$  is a real number). Thus  $p_3(s; \tau, \theta, \phi)$  has at most one jump discontinuity at  $s = \tau$  which happens only when  $|x_3(\tau)| = 1$ . The

explicit form of extremal  $v(s)$  can now be expressed as

$$\begin{aligned} -v(s) &= \frac{2.5}{\pi} \operatorname{sgn} [p_3(s; \tau, \theta, \phi)] \\ &= \frac{2.5}{\pi} \operatorname{sgn} [p_3(s; \tau, \theta, \phi)/\cos \phi] . \end{aligned}$$

Furthermore, the extremal  $v(s)$  is also the time-optimal  $v(s)$  by Russell's sufficiency condition [1].

In Figure 4, the function  $p_3$  for the typical case is also sketched. The formulas for parameters  $\tau_i$ ,  $i = 1, 2, \dots, 2N$ ,  $\delta$ , and  $\phi_0$  are determined for all possible choices of  $-\pi \leq \theta \leq 0$ ,  $|\phi| < \pi/2$ , and  $0 \leq \tau < \infty$ . All the results are tabulated in Charts I and II. To use these charts, first locate the Case Number from Chart I for the appropriate ranges of  $\tau$  and  $\theta$ . Then on Chart II, for every Case Number and every range of  $\phi$ , a set of parameters of  $\tau_i$ ,  $i = 1, 2, \dots, 2N$ ,  $\delta$ , and  $\phi_0$  are given.

#### 4. PLAN OF FUTURE WORK

An immediate step is to extend the study of the harmonic oscillator to the cases where the ratio of control amplitude limit to the control rate limit assumes various numbers. At the present time, the investigation is almost completed for this ratio ranging from  $2\pi/5$  to  $2\pi/4$ . Although the range seems small, the complexity of the problem increases significantly because of the interaction between extremal control amplitude and extremal control rate.

As outlined in the previous report [3], a study of an underdamped oscillatory plant with bounded amplitude and rate control will be carried out next. Since this represents a process in which the plant

has a pair of complex conjugate roots, the problem is relatively important in the practical sense. Other major plans are listed in the previous report and will not be restated here.

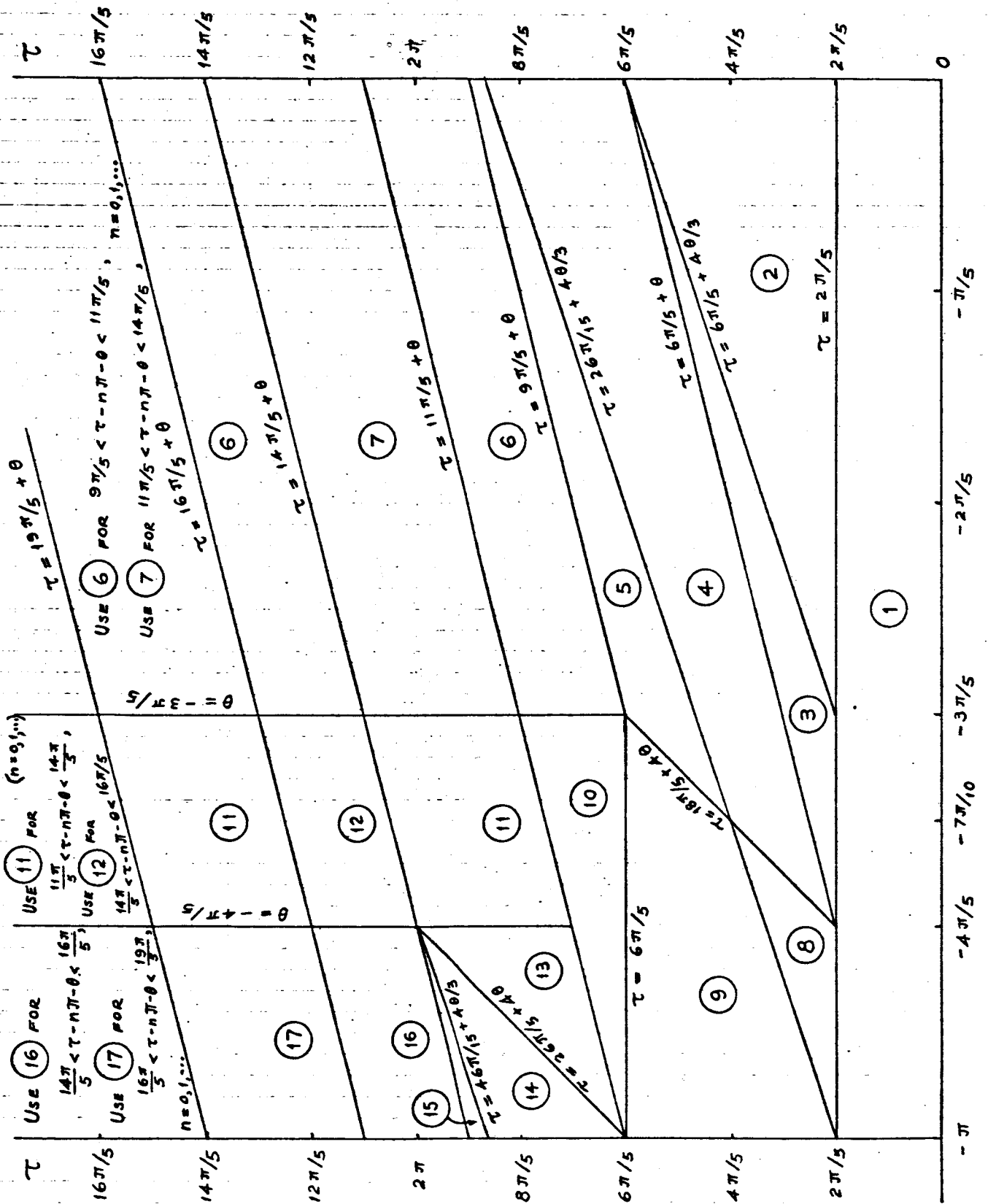


CHART I

K=2.5	<div style="border: 1px solid black; padding: 5px; text-align: center;"> <math>\phi</math>  <math>\tau_1, \tau_2</math>  <math>\delta, \phi_0</math>  CASE </div>	$n/2 > \phi > 0$		$0 > \phi > \phi_2$	$\phi_2 > \phi > \phi_6$	$\phi_6 > \phi > -n/2$
		$\phi_2 = -\tan^{-1}[\cos\theta - \cos(\pi/2 - \phi)]$	$0 > \phi > \phi_1$	$\phi_1 = -\tan^{-1}[\cos\theta - \cos(-\theta + \pi - \frac{2n}{5})]$	$\phi_2 > \phi > \phi_6$	$\phi_4 > \phi > -n/2$
1		$\tau_1 = \tau_{2N} = 0$	$N=1$	$\delta=0$		
2	$\tau_1 = \frac{2n}{5}$ $\tau_{2N} = \tau, N=1$ $\delta=1, \phi_0=0$	$\tau_1 = \frac{2n}{5}$ $\tau_{2N} = -\theta + \tau - \pi/5 [\cos\theta + \tan\phi], N=1$ $\delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=1, \phi_0 = \phi_4$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=1, \phi_0 = \phi_6$
3						
4	$\tau_1 = \frac{4}{3}(\tau - \theta) - \frac{6n}{5}$ $\tau_{2N} = \tau, N=1$ $\delta=1, \phi_0=0$	$\tau_1 = \frac{4}{3}(\tau - \theta) - \frac{6n}{5}$ $\tau_{2N} = -\theta + \tau - \pi/5 [\cos\theta + \tan\phi]$ $N=1, \delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=0$	$N=1$	
5						
6	$\tau_1 = \frac{2n}{5}$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = \tau, \delta=1, \phi_0=0$	$\tau_1 = \frac{2n}{5}$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - \cos^{-1}[\cos\theta + \tan\phi], \delta=0$	$\tau_1 = \frac{2n}{5}$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - \cos^{-1}[\cos\theta + \tan\phi], \delta=0$	$\tau_1 = \frac{2n}{5}$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - \cos^{-1}[\cos\theta + \tan\phi], \delta=0$	$\tau_1 = \frac{2n}{5}$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = \tau - \frac{4}{3}(\pi + \theta), \delta=1, \phi_0 = \phi_5$	$\tau_1 = \frac{2n}{5}$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = \tau - \frac{4}{3}(\pi + \theta), \delta=1, \phi_0 = \phi_5$
7	$\tau_1 = \frac{4}{3}(\tau - \theta) - (\frac{32}{15} + \frac{4}{3}n)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = \tau, \delta=1, \phi_0=0$	$\tau_1 = \frac{4}{3}(\tau - \theta) - (\frac{32}{15} + \frac{4}{3}n)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - \cos^{-1}[\cos\theta + \tan\phi], \delta=0$	$\tau_1 = \frac{4}{3}(\tau - \theta) - (\frac{32}{15} + \frac{4}{3}n)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - \cos^{-1}[\cos\theta + \tan\phi], \delta=0$	$\tau_1 = \frac{4}{3}(\tau - \theta) - (\frac{32}{15} + \frac{4}{3}n)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - \cos^{-1}[\cos\theta + \tan\phi], \delta=0$	$\tau_1 = \frac{4}{3}(\tau - \theta) - (\frac{32}{15} + \frac{4}{3}n)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = \tau - \frac{4}{3}(\pi + \theta), \delta=1, \phi_0 = \phi_5$	$\tau_1 = \frac{4}{3}(\tau - \theta) - (\frac{32}{15} + \frac{4}{3}n)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, i=1, \dots, n-1$ $\tau_{2(n+1)} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = -\theta + \tau - (\frac{2}{5} + n + i - 2)\pi, N=n+2$ $\tau_{2N} = \tau - \frac{4}{3}(\pi + \theta), \delta=1, \phi_0 = \phi_5$
K=2.5	<div style="border: 1px solid black; padding: 5px; text-align: center;"> <math>\phi</math>  CASE </div>	$0 > \phi > \phi_3$	$\phi_3 > \phi > \phi_5$	$\phi_5 > \phi > -\frac{n}{2}$		
		$\phi_3 = -\tan^{-1}[\cos\theta - \cos(\frac{2n}{5})]$	$\phi_5 = -\tan^{-1}[\cos\theta - \cos(\frac{2n}{5})]$	$\phi_5 > \phi > -\frac{n}{2}$		

## CHART II

K=2.5	<div><div><div><math>\phi</math></div><div><math>\tau_1, \tau_2, \dots</math> <math>\delta, \phi_0</math></div></div><div>CASE</div></div>	$\pi/2 > \phi > \phi_4 (> 0)$ $\phi_4 = +\tan^{-1} [\cos(\theta + \pi/2 - \pi/2) - \cos\theta]$	$\phi_4 > \phi > \phi_6$ $\phi_6 = -\tan^{-1} [\cos\theta \cdot \cos(\pi/2 - \pi/4)]$	$\phi_6 > \phi > -\pi/2$
		$\phi_4 > \phi > \phi_1$ $\phi_1 = -\tan^{-1} [\cos\theta \cdot \cos(\theta + \pi/2)]$		
8		$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=1, \phi_0 = \phi_4$	$\tau_1 = \tau_{2N} = 0$ $N=1, \delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=1, \phi_0 = \phi_6$
9			$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=0$	$\tau_1 = \tau_{2N} = 0$ $N=1$ $\delta=1, \phi_0 = \phi_5$
10		$\tau_1 = \tau_{2N} = \tau - \frac{4\pi}{5}, N=1$ $\delta=1, \phi_0 = \phi_{13}$	$\tau_1 = \tau_{2N} = -\theta + \tau + \cos^{-1} [\cos\theta + \tan\phi] - 2\pi$ $N=1, \delta=0$	
11		$\tau_1 = \frac{4\pi}{5}(\tau - \theta) - (\frac{3\pi}{5} + \frac{4}{5}\pi)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2\pi}{5} + \pi + i - i)\pi, i=1, \dots, N-1$ $\tau_{2i+1} = -\theta + \tau - (\frac{2\pi}{5} + \pi + i)\pi, N=\pi+1$ $\tau_{2N} = \tau - 4\pi/5, \delta=1, \phi_0 = \phi_{13}$	$\tau_1 = \frac{4\pi}{5}(\tau - \theta) - (\frac{3\pi}{5} + \frac{4}{5}\pi)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2\pi}{5} + \pi + i - i)\pi, i=1, \dots, N-1$ $\tau_{2i+1} = -\theta + \tau - (\frac{2\pi}{5} + \pi + i - i)\pi, N=\pi+1$ $\tau_{2N} = -\theta + \tau + \cos^{-1} [\cos\theta + \tan\phi] - 2\pi, \delta=0$	$\tau_1 = \frac{4\pi}{5}(\tau - \theta) - (\frac{3\pi}{5} + \frac{4}{5}\pi)\pi$ $\tau_{2i} = -\theta + \tau - (\frac{2\pi}{5} + \pi + i - i)\pi, i=1, \dots, N-1$ $\tau_{2i+1} = -\theta + \tau - (\frac{2\pi}{5} + \pi + i - i)\pi, N=\pi+1$ $\tau_{2N} = \tau - 4\pi/5(\pi + \theta), \delta=1, \phi_0 = \phi_5$
12		$\tau_1 = \tau_{2N} = \tau - \frac{4\pi}{5}, \delta=1, \phi_0 = \phi_{13}$	$\tau_1 = \tau_{2N} = -\theta + \tau + \cos^{-1} [\cos\theta + \tan\phi] - 2\pi, \delta=0$	$\tau_1 = \tau_{2N} = \tau - 4\pi/5(\pi + \theta), \delta=1, \phi_0 = \phi_5$
K=2.5	<div><div><div><math>\phi</math></div><div><math>\tau_1, \tau_2, \dots</math> <math>\delta, \phi_0</math></div></div><div>CASE</div></div>	$\pi/2 > \phi > \phi_{13} (> 0)$ $\phi_{13} = +\tan^{-1} [\cos(\frac{6\pi}{5} + \theta) - \cos\theta]$	$\phi_{13} > \phi > \phi_5$ $\phi_5 = -\tan^{-1} [\cos\theta - \cos(\frac{2\pi}{3} - \frac{\theta}{3})]$	$\phi_5 > \phi > -\pi/2$

CHART II (CONTINUED)

[illegible]

## CHART II (CONTINUED)

## PART D

### STOCHASTIC OPTIMAL CONTROL

#### 1. INTRODUCTION

During the coordination meeting at Purdue on March 11, 1966, the antenna pointing problem was discussed to some extent. It was agreed that the control problem should be treated as a steady state problem or its equivalent. The block diagram of the system is shown in Figure 5 in which the antenna on a space vehicle is subject to random disturbances.

To formulate the problem meaningfully, it is necessary to select a performance index having a physical significance. Section 2 discusses the motivation of the problem formulation. The performance index which is practical in the engineering sense, yet is mathematically tractable, is then introduced. With respect to the performance index, the optimal control problem is thus defined. Section 3 gives a brief summary of technical progress during the reporting period. Detailed discussions are presented in the Appendix. Section 4 outlines the plan of future work.

#### 2. FORMULATION OF ANTENNA POINTING PROBLEM

In the antenna system, the basic purpose is the transmission of information. A logical performance index, then, is the expected information rate. Such a criterion has a close connection to the pointing accuracy which is measured by the signal strength. Let

$r$  = information rate,

$s$  = signal strength,

$p$  = pointing accuracy,

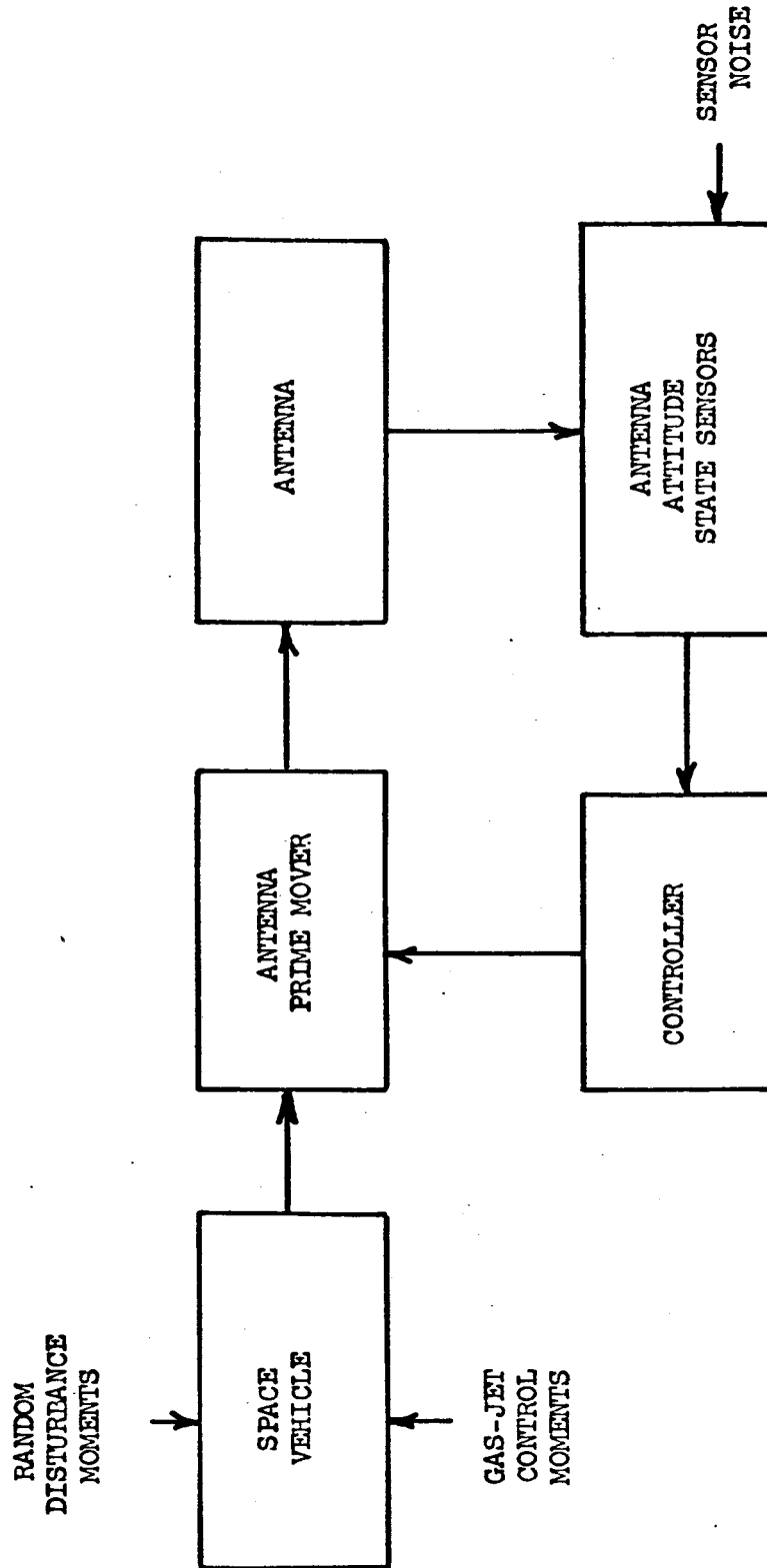


FIGURE 5

then

$$r = f(s) = f(g(p)) = h(p)$$

Equivalently,

$$R = H(E)$$

where  $R$  = error rate, and  $E$  = pointing error. The function  $H$ , then, is a measure of accuracy, and hence is a logical performance index to be minimized in the process of determining an optimal controller.

Typically,  $H$  can be sketched as in Figure 6 in which the measure of performance is graphically classified into three regions, viz., good, transition, and poor. These regions are directly connected to the antenna pointing direction. This is shown in Figure 7 in the  $(\theta, \phi)$ -coordinates where  $\theta$  and  $\phi$  is the coordinate system of the pointing direction.

With the measure of performance so defined, the controller is assigned to operate in two modes as follows:

Mode 1 - When the pointing angle is inside the good performance region, the controller generates a control signal which minimizes the probability of exiting from that region during some time interval  $T_h$ .

Mode 2 - When the pointing angle is in the poor performance region, the controller generates a control signal which maximizes the probability of entering the good performance region during some time interval  $T_a$ .

Thus the optimization procedure can be carried out in two separate parts:

- (1) In each mode of operation, determine the control signal that

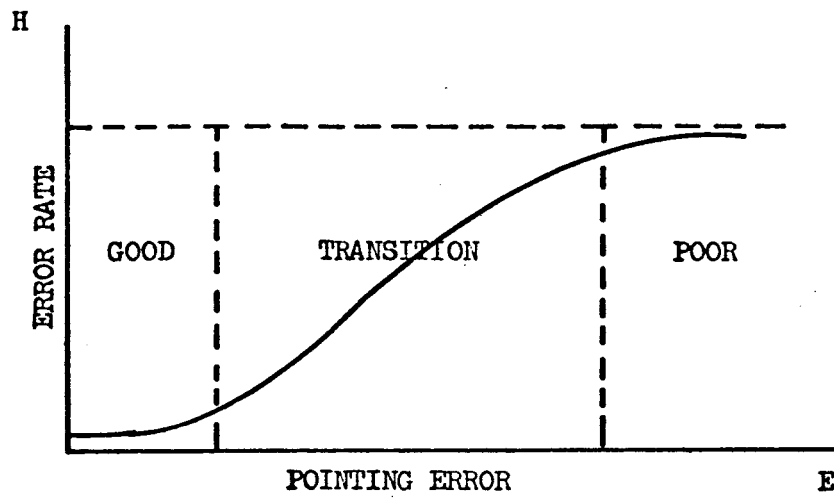


FIGURE 6

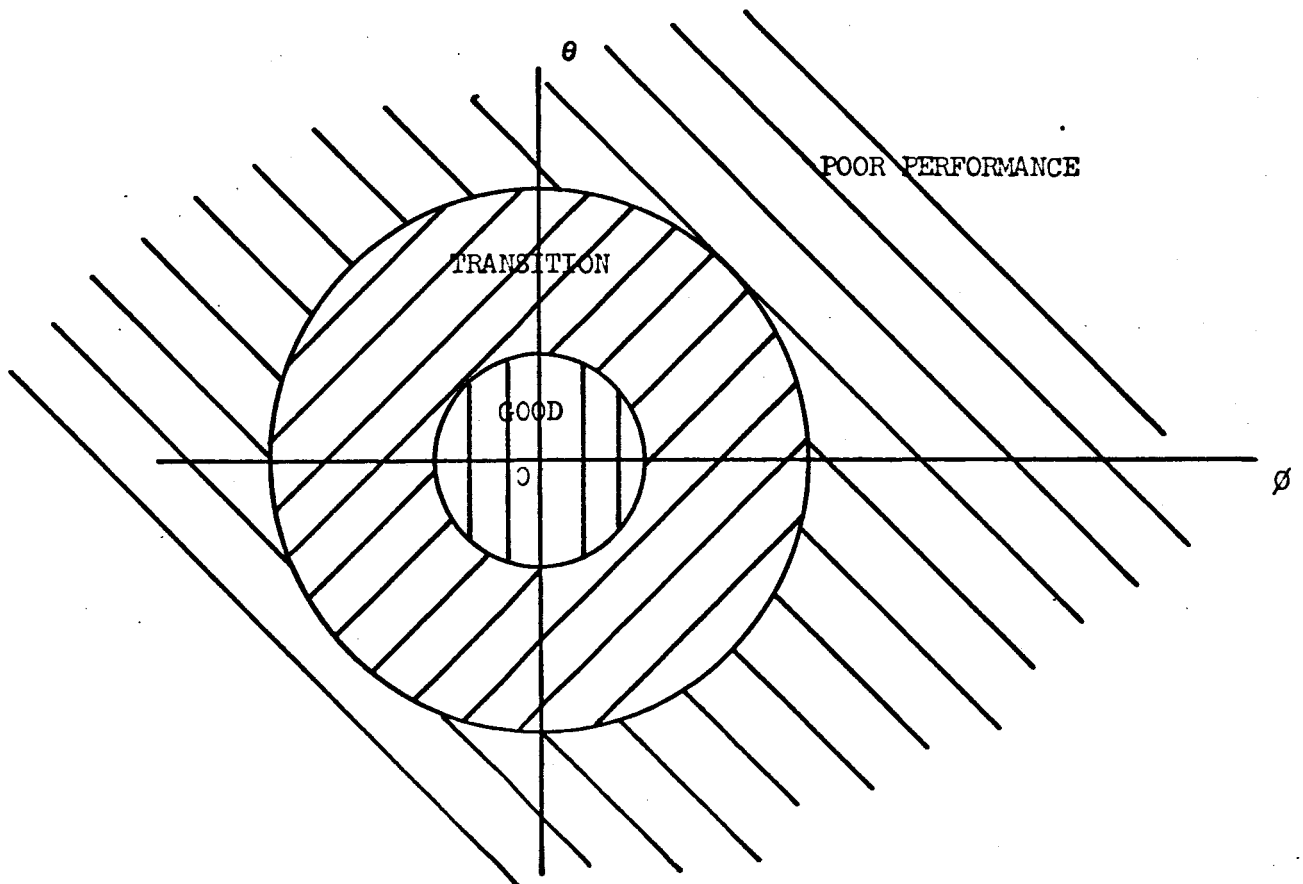


FIGURE 7

minimizes (or maximizes) the appropriate probability.

(2) Minimize the function  $H$  (representing the error rate) with respect to the sizes of transition and good performance regions (the radii of the boundary circles of these regions), and to  $T_h$  and  $T_a$ .

### 3. SUMMARY OF TECHNICAL PROGRESS

During the reporting period, effort was directed toward the determination of the control signal which minimizes (or maximizes) the relevant probability in each mode of operation. For a linear system with white noise disturbances, the method of determining the control signal was discussed in the previous progress report [3]. The results of the extended study of the problem are summarized in this section with the detailed derivation given in the Appendix.

The block diagram of the system under investigation is shown in Figure 8 in which  $x$  is the system state,  $n$  is the random disturbance and  $u$  represents control. For the purpose of discussion, assume the state is in a situation such that the control  $u$  is in Mode 2 of operation. Let  $\Psi(t_1, x_1, t_1 + T_a)$  be the probability of entering the good performance region in the time interval  $[t_1, t_1 + T_a]$ , then  $\Psi$  can be found as the solution of a partial differential equation of a boundary value problem. A first order approximation to  $\Psi$  in the  $(x_1, t_1, T)$  - space is known [2]. For the purposes of determining the optimal control  $u$ , it only requires determining  $\Psi(0, x_0, T_a)$  as an implicit function of  $u$ . An iterative scheme of finding optimal  $u$  by optimizing  $\Psi$  is discussed in the Appendix. The method can be applied to systems with no restrictions on their order, and hence has an advantage over the scheme given by Mishchenko [2].

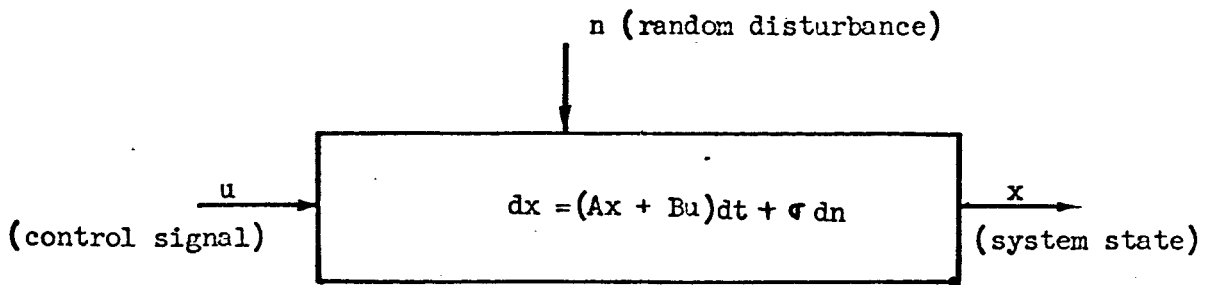


FIGURE 8

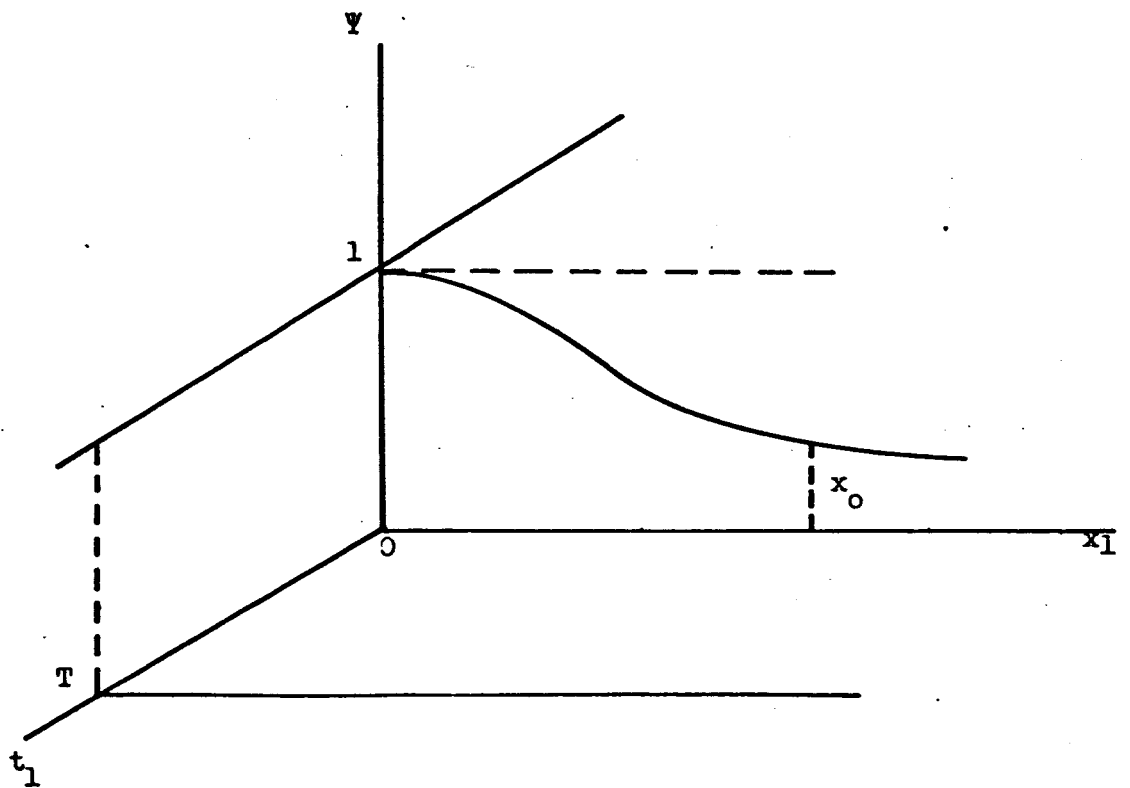


FIGURE 9

#### 4. PLAN OF FUTURE WORK

The simulation of the antenna pointing system on a digital computer will be the next goal in the plan. This involves the writing of a computer problem based on the flow chart discussed in the Appendix. The convergence problem of the computational scheme will also be investigated.

The extension of the method to higher order systems will be examined closely. Attention will be focused on any hidden pitfalls. Once this is completed, numerical data for a physical space vehicle will be used as a test model for the computational method. A comparison of the results so obtained against those from existing control systems is also planned.

# APPENDIX

The antenna problem as a "pursuit" problem was discussed to some extent in the previous progress report [3]. The solution of the problem for first order systems has been examined in more detail during this reporting period, and is presented in the following paragraphs. The extension of this technique to high order systems is in progress.

Consider the system which is described by the stochastic differential equation

$$dx = A(x, t) dt + u(t) dt + \sigma(x, t) dn \quad (1)$$

$$x(0) = x_0$$

where

$x$  = system state

$u$  = control signal

$n$  = a Brownian motion process

Doob [4] showed that under suitable restrictions on  $A$  and  $\sigma$ ,

$$\frac{\partial p}{\partial t_1} + [A(x_1, t_1) + u(t_1)] \frac{\partial p}{\partial x_1} + \frac{1}{2} \sigma^2(x_1, t_1) \frac{\partial^2 p}{\partial x_1^2} = 0 \quad (2)$$

where  $p(t_1, x_1, t_2, x_2)$  = probability that  $x(t_2) < x_2$  given that  $x(t_1) = x_1$ . Mishchenko [2] showed that if the statistics of the  $x$ -process are described by (2), then

$$\frac{\partial \Psi}{\partial t_1} + [A(x_1, t_1) + u(t_1)] \frac{\partial \Psi}{\partial x_1} + \frac{1}{2} \sigma^2(x_1, t_1) \frac{\partial^2 \Psi}{\partial x_1^2} = 0 \quad (3)$$

with boundary conditions (see Figure 9)

$$\Psi(t_1, 0, T) = 1 \text{ for all } t_1, \quad (4)$$

and

$$\Psi(T, x_1, T) = 0 \text{ for } x_1 \neq 0 \quad (5)$$

where  $\Psi(t_1, x_1, T)$  = probability that  $[x(\tau) = 0]$  for some  $\tau \in [t_1, T]$ .

To maximize  $\Psi(0, x_0, T)$  with respect to  $u$ , a numerical method for determining  $\Psi$  for a given  $u$  is given first. The initial step in the procedure is to guess  $\Psi(0, x_1, T)$ . That is, let

$$\Psi(0, x_1, T) = \varphi_0(x_1) \quad (6)$$

where  $\varphi_0$  is an initially guessed function. Then compute  $(\partial\Psi/\partial t_1)|_{t_1=0}$

by using equations (3) and (6). Thus, by the relation

$$\Psi(t_1 + \Delta t, x_1, T) = \Psi(t_1, x_1, T) + \frac{\partial\Psi}{\partial t_1}(t_1, x_1, T) \Delta t, \quad (7)$$

where

$$\Delta t = T/M, \Psi(\Delta t, x_1, T)$$

can also be computed.

Using the same procedure, compute  $\Psi(2\Delta t, x_1, T)$ , etc. The process is repeated until  $\Psi(M\Delta t, x_1, T)$  is obtained, say,

$$\Psi(M\Delta t, x_1, T) = \varphi_T(x_1) \quad (8)$$

If  $\varphi_T(x_1) = 0$  for all  $x_1 \neq 0$ , then  $\varphi_0(x_1)$  correctly represents  $\Psi(0, x_1, T)$ . It is clear, then, that the functional

$$I_1(\varphi_0) = \int_0^{x_0} [\varphi_T(x_1)]^2 dx_1 \quad (9)$$

is minimal for  $\varphi_0(x_1) = \Psi(0, x_1, T)$ . Thus the problem becomes the minimization of  $I_1(\varphi_0)$  with respect to  $\varphi_0(x_1)$ . This minimization problem in its discretized version is presently carried out by a gradient technique as follows:

Let the function  $\varphi_0(x_1)$  be represented by a vector  $\underline{\varphi_0}$  whose components are

$$\varphi_0^1 = \varphi_0(x_1^1)$$

with

$$0 = x_1^1 < x_1^2 < \dots < x_1^N = x_0.$$

Let the function  $u(t_1)$  be represented by a vector  $\underline{u}$  whose components are

$$u^1 = u(t_1^1)$$

with

$$0 = t_1^1 < \dots < t_1^N = T .$$

The solution to the discretized version of the optimization problem will be carried out in the  $2N$ -dimensional space  $\{u^1, u^2, \dots, u^N, \phi_o^1, \dots, \phi_o^N\}$ . A flow chart of the computational algorithm is shown in Figure 10. The procedure begins with choosing  $\underline{u}_o$  for  $\underline{u}$  and  $\phi_{oo}$  for  $\phi_o$ , as shown in Steps 1 and 2. Step 3 computes  $\phi_T$  and  $I_1(\phi_o)$  by the equations (3) through (9). Step 4 tests if  $\phi_o$  has been determined (whether from initial guess or successive iterations) with sufficient accuracy. If not, its value is improved as follows:

Let  $\underline{v}$  be a  $2N$ -dimensional unit vector normal to the  $\underline{u} = \underline{u}_o$  manifold at  $\{\underline{u}_o, \phi_{oo}\}$ , then the iterated  $\phi_{ol}$  of  $\phi_{oo}$  can be computed from

$$\{\underline{u}_o, \phi_{ol}\} = \{\underline{u}_o, \phi_{oo}\} - h_1 \underline{v} \langle \underline{v}, \nabla I_1 \rangle$$

where

$$\underline{v} = \left\{ \frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^N}, \frac{\partial}{\partial \phi_o^1}, \dots, \frac{\partial}{\partial \phi_o^N} \right\},$$

$h_1$  = step size factor,

$\langle , \rangle$  denotes the inner product .

This computational task is rendered in Step 5.

When Step 4 indicates that  $\phi_o$  has reached a value with a desired accuracy, the improvement of  $\underline{u}$  begins as follows:

Let  $\underline{w}$  be a  $2N$ -dimensional unit vector normal to  $\nabla I_1$  at  $\{\underline{u}_o, \phi_{oo}\}$ .

Then

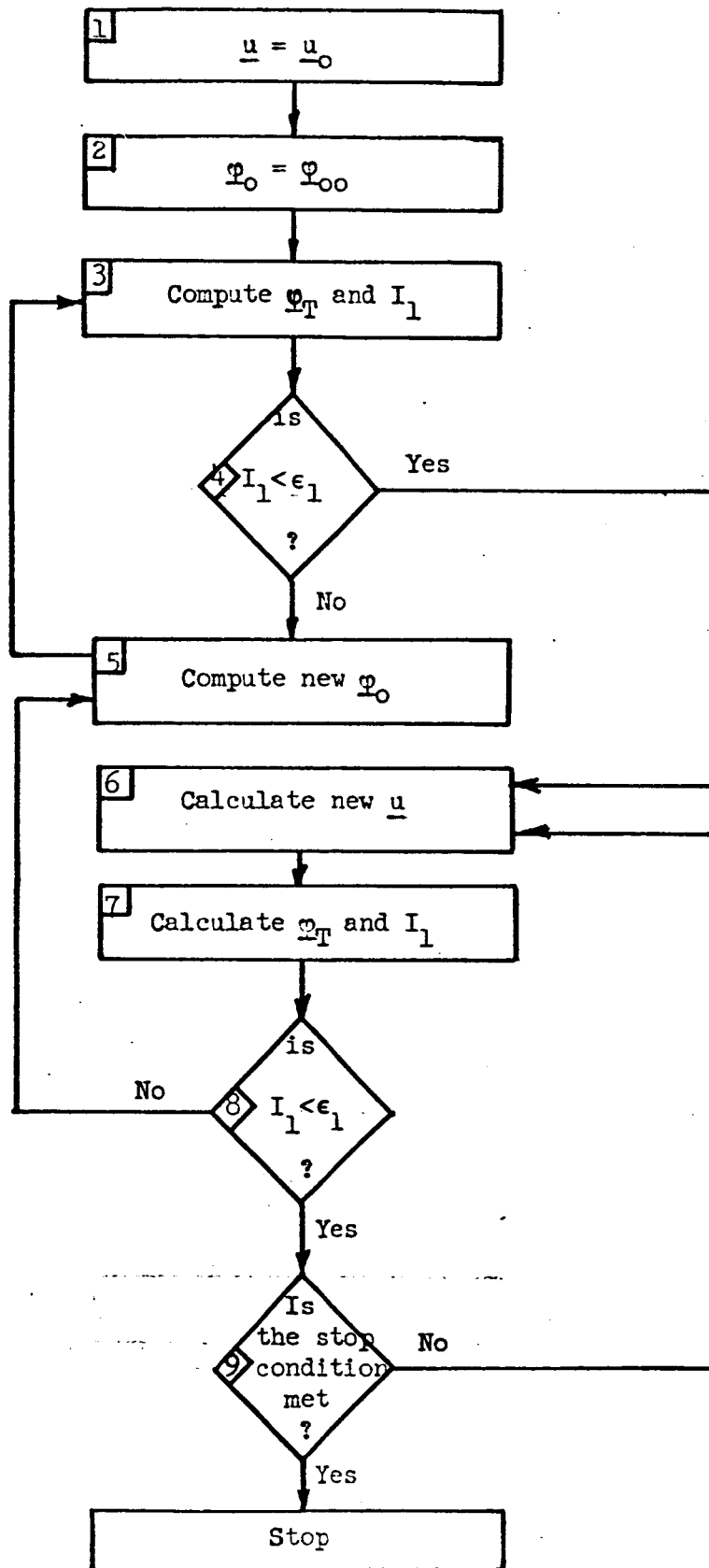


FIGURE 10

$$\{ \underline{u}_1, \underline{\phi}_{01} \} = \{ \underline{u}_0, \underline{\phi}_{00} \} + h_2 \underline{W} < \underline{W}, \nabla \phi_0^N >$$

where  $h_2$  = step size factor. This computational procedure is carried out in Step 6, which has the following meaning:  $\{ \underline{u}_0, \underline{\phi}_{00} \}$  is being incremented in the direction which increases  $\phi_{00}^N$  but leaves  $I_1$  unaltered. If  $I_1$  is indeed unaltered, then  $\phi_{01}^N = \Psi(0, x_0, T)$ . Because of discretization error,  $I_1$  will in fact be changed somewhat. For this reason, Steps 7 and 8 are inserted to test the magnitude of the discretization errors introduced in Step 6. If  $I_1$  is not within the desired accuracy, these errors are corrected by returning to Step 5. Otherwise, the computation is terminated if a prescribed number of improvement cycles (which gives a stop condition) is completed.

REFERENCES

1. Luh, J. Y. S., D. L. Russell, E. B. Lee, Bounded Phase - coordinate Control, Final Technical Report to NASA Contract NAS w 986, Honeywell Document No. 12006 - FRI, July 1965, Chapter 3.
2. Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience, 1962, Chapter VI and VII.
3. Luh, J. Y. S., G. E. O'Connor, and J. S. Shafran, Investigation of Optimization of Attitude Control Systems, Quarterly Progress Report for Period 15 Sept. through 31 Dec. 1965 for JPL, Contract No. 950670.
4. Doob, J. L., Stochastic Processes, Wiley, New York, 1953.